Renormalization-group theoretical reduction

K. Nozaki[1] and Y. Oono[2,3]
[1]*Department of Physics, Nagoya University, Nagoya, Japan*
[2]*Research Institute of Mathematical Sciences, Kyoto University, Kyoto, Japan*
[3]*Department of Physics, University of Illinois at Urbana-Champaign, 1110 W. Green Street, Urbana, Illinois 61801-3080**
(Received 17 April 2000; revised manuscript received 29 August 2000; published 19 March 2001)

It has been known for some time that singular perturbation and reductive perturbation can be unified from the renormalization-group (RG) theoretical point of view. However, renormalization-group approaches to singularly perturbed problems require explicit perturbation results, so they could be complicated practically. The approach proposed in this paper has considerably eliminated the need for explicit perturbation results, making the RG approach simpler than many conventional singular perturbation approaches. Consequently, we may assert that reductive extraction of global features of the problem is the essence of singular perturbation methods.

DOI: 10.1103/PhysRevE.63.046101 PACS number(s): 64.60.Ak, 45.70.Qj, 02.30.Mv, 05.10.Cc

I. INTRODUCTION

It has been realized for some time [1] that renormalization-group equations associated with the field theoretical renormalization-group (RG) methods [Stückelberg-Petermann(-Gell-Mann-Low) RG] can be interpreted as slow motion equations, amplitude equations, and so forth. These equations have been traditionally derived with the aid of the so-called reductive perturbation theory [2,3]. It is true that all the famous named equations, such as the Burgers equation, the Kuramoto-Sivashinsky equation, the Boltzmann equation [4], the nonlinear Schrödinger equation, etc., can be interpreted as RG equations.

Suppose a spatially extended system exhibits a state; it need not be simple, but practically it is often a uniform, or a space-time periodic state. We add to the system a perturbation, e.g., some nonlinear terms, dissipative terms, etc. The system is usually not structurally stable against such perturbations, so that perturbation results are, if computed naively, plagued by singularities (secular terms). It has been recognized that these singularities in the naive perturbation theories can be renormalized away by the modification (renormalization) of the parameters in the unperturbed state (amplitude, phase, etc.) [5–7] and the results agree with or are sometimes (numerically) better than those traditionally computed with the aid of singular perturbation methods [8]. The modified parameters are governed by the RG equations that turn out to be, e.g., large-scale slow-motion equations (reduced equations) (often the famous named equations). Let us call the method to obtain space-time large scale equations as RG equations the *RG theoretical reduction* (or the *reductive RG method*).

Graham [9] has utilized this RG theoretical reduction to derive the isotropic Newell-Whitehead equation [10] proposed by Gunaratne *et al.* [11]. The derivation was questioned by Matsuba and Nozaki [12]: while the derivation is based on explicit elimination of secular terms, it is not clear whether all of the secular terms are consistently removed and whether differential operators with different orders are consistently retained in the derived equation.

As can be seen from Graham's paper, Sasa's paper [13], or Ref. [8], it appears that we can accomplish by the RG reduction the same goal the conventional reductive perturbation can achieve. However, the RG reduction seems to be more demanding in the actual calculation than the conventional reductive perturbation [3], because the former seems to require explicit functional forms of secular terms. Since we are often interested in deriving the reduced equations, and not in their explicit solutions, the RG reduction so far being practiced [1,8] is methodologically awkward.

The purpose of the present paper is to free the RG theoretical reduction from the necessity of explicit secular terms as much as possible, and to stress the significance of system reduction in singular perturbation in general. The conventional reductive perturbations do not require explicit perturbation results. This advantageous feature can now be combined with advantages of the RG approach demonstrated in Ref. [8]. For example, to the lowest nontrivial order, that is, to the order usual calculations are performed, our reductive RG method is probably the simplest reduction method. As [8] this paper largely consists of demonstrations with examples; it is still hard to assert anything extremely general, but the approach described in this paper can handle the examples in Ref. [8] that already cover many representative examples of classic singular perturbation problems.

In Sec. II, details of the second order standard RG perturbative calculation are supplied for the Rayleigh equation to motivate the reductive RG theory proposed in this paper. The practically oriented reader can skip this section. In Sec. III, our improved method for autonomous ordinary differential equations (ODE's) is explained. The method introduces a device called the proto-RG equation that can be obtained almost by inspection. Its reduction to the standard RG equation is, when it can be obtained by our conventional RG method, a purely mechanical calculation much easier than explicitly obtaining the secular terms. In Sec. IV, the RG reduction to all orders is performed for linear ODE's. In Sec. V, we briefly discuss ''beyond all orders.'' We demonstrate that renormalized perturbation series in principle can contain

*Permanent address.

all the information about the original system, so that such as $e^{-t/\epsilon}$ can be obtained from the power series expansion in ϵ. In Sec. VI, we note some important features of partial differential equations (PDE's). In Sec. VII, as the simplest application of reductive RG to PDE's, we discuss the reduction of some examples of evolution equations. In Sec. VIII, the reduction of the Swift-Hohenberg equation is revisited to the second order. In Sec. IX concluding remarks are given.

II. STANDARD RG PERTURBATION REVISITED

To understand the RG reduction scheme and to understand the merit of the streamlined version explained in this paper, or at least to motivate the idea, it is advantageous to have some experience of the RG approach to the singular perturbation. Therefore we give nontrivial details of an example whose final result has already been published in Ref. [8]. For those who wish to know the practical aspect of the proposed method in this paper, the first few paragraphs of the next section should be a good starting point [with reference to the formulas in this section up to Eqs. (2.6) and (2.11)].

The example we discuss is the Rayleigh equation:

$$\frac{d^2y}{dt^2}+y=\epsilon\frac{dy}{dt}\left[1-\frac{1}{3}\left(\frac{dy}{dt}\right)^2\right]. \tag{2.1}$$

We expand the solution as

$$y=y_0+\epsilon y_1+\epsilon^2 y_2+\cdots. \tag{2.2}$$

These expansion coefficients obey the following equations:

$$\left(\frac{d^2}{dt^2}+1\right)y_0=0, \tag{2.3}$$

$$\left(\frac{d^2}{dt^2}+1\right)y_1=\frac{dy_0}{dt}-\frac{1}{3}\left(\frac{dy_0}{dt}\right)^3, \tag{2.4}$$

$$\left(\frac{d^2}{dt^2}+1\right)y_2=\frac{dy_1}{dt}\left[1-\left(\frac{dy_0}{dt}\right)^2\right]. \tag{2.5}$$

Let us write the solution to Eq. (2.3) as

$$y_0=Ae^{it}+A^*e^{-it}, \tag{2.6}$$

where A is a complex constant. Equation (2.4) reads

$$\left(\frac{d^2}{dt^2}+1\right)y_1=iA(1-|A|^2)e^{it}+\frac{i}{3}A^3e^{3it}+\text{c.c.}, \tag{2.7}$$

where c.c. denotes the complex conjugate terms. We need not worry about the initial condition for this equation, so we have only to make a special solution

$$y_1=\frac{1}{2}te^{it}A(1-|A|^2)-\frac{i}{24}A^3e^{3it}+\text{c.c.} \tag{2.8}$$

The right hand side (rhs) of Eq. (2.5) need not be computed in detail, but we must keep the resonant and unbounded terms. Thus the rhs of Eq. (2.5) reads

$$\begin{aligned}&\frac{1}{2}(1-2|A|^2)\{A(1-|A|^2)+itA(1-|A|^2)\}e^{it}\\&\quad+\frac{1}{8}A|A|^4e^{it}+\frac{1}{2}|A|^2\{A(1-|A|^2)-iA(1-|A|^2)t\}e^{it}\\&\quad+\frac{i}{2}A^3(1-|A|^2)te^{3it}+\text{c.c.}\end{aligned} \tag{2.9}$$

From this the secular terms (indicated by $[\,]_S$) in y_2 read

$$\begin{aligned}[y_2]_S=&\frac{1}{2}A(1-|A|^2)^2\frac{t}{2i}e^{it}+\frac{i}{2}A(1-|A|^2)(1-3|A|^2)\\&\times\left(\frac{t^2}{4i}+\frac{t}{4}\right)e^{it}+\frac{1}{16i}A|A|^4te^{it}\\&-\frac{i}{16}A^3(1-|A|^2)te^{3it}+\text{c.c.}\end{aligned} \tag{2.10}$$

The basic strategy of renormalization of secular terms is to split the secular time dependence such as t^2 in the prefactors of the exponential terms as $t^2=(t^2-\tau^2)+\tau^2$, and absorb τ^2 into the renormalization constant Z of the free parameter in the solution (for the present example A is the parameter we can choose freely by choosing an appropriate initial condition; note that the renormalization procedure does not necessarily imply the time shift $t\to t-\tau$).

We introduce the renormalized amplitude A_R as $A=ZA_R$. We expand Z in powers of ϵ and write

$$A=A_R(1+\epsilon Z_1+\epsilon^2Z_2+\cdots). \tag{2.11}$$

The equation for the renormalized amplitude should be the amplitude equation

$$\frac{dA}{d\tau}=0. \tag{2.12}$$

From this to order ϵ^2 we obtain the renormalization-group equation

$$\frac{dA_R}{d\tau}=-\epsilon A_R\left(\frac{dZ_1}{d\tau}-\epsilon Z_1\frac{dZ_1}{d\tau}+\epsilon\frac{dZ_2}{d\tau}\right). \tag{2.13}$$

Thus we have only to determine the renormalization constant.

Z is determined to absorb the powers of τ separated out from the secular t dependences in the bare perturbation result for y. The first order is easy to obtain from Eq. (2.8) as

$$Z_1+(1-|A_R|^2)\frac{\tau}{2}=0. \tag{2.14}$$

To order unity this gives (note that $dA_R/d\tau$ is of order ϵ)

$$\frac{dZ_1}{d\tau} = -\frac{1}{2}(1-|A_R|^2), \tag{2.15}$$

so that the renormalization-group equation to order ϵ reads [cf. Eq. (2.13))]

$$\frac{dA_R}{d\tau} = \frac{\epsilon}{2}A_R(1-|A_R|^2). \tag{2.16}$$

The second order singular terms contain terms proportional to te^{it}, t^2e^{it}, and te^{3it} (and their complex conjugates). The singular terms containing e^{3it} are totally removed by Z_1 just determined as Eq. (2.14). If we wish to compute higher order results, then we must deal with e^{5it}, e^{7it}, etc. The secular terms containing these exponential factors (henceforth called *nonresonant secular terms*) are all removed by the results for the renormalization constant obtainable through lower order calculations as in this case (for example, to the second order, Z_2 is not needed, but only Z_1 as seen above). This tells us that if we wish to determine Z, we have only to study secular terms containing $e^{\pm it}$, which we will henceforth call the *resonant secular terms*.

To remove the resonant secular terms to order ϵ^2, we must require [14]

$$Z_2 = \frac{\tau^2}{8}(1-|A_R|^2)(1-3|A_R|^2) + \frac{\tau}{8}i(1-|A_R|^4) + \frac{i}{16}|A_R|^4\tau. \tag{2.17}$$

To obtain the amplitude equation we need the derivative of Z_1 with respect to τ to order ϵ:

$$\frac{dZ_1}{d\tau} = -\frac{1}{2}(1-|A_R|^2) + \epsilon\frac{\tau}{2}|A_R|^2(1-|A_R|^2), \tag{2.18}$$

where Eq. (2.16) has been used. Putting the results so far obtained into Eq. (2.13), we see total cancellation of the terms proportional to τ, and obtain

$$\frac{dA_R}{d\tau} = \epsilon\frac{1}{2}A_R(1-|A_R|^2) - \epsilon^2\frac{i}{16}A_R(2-|A_R|^4). \tag{2.19}$$

This is equivalent to the RG equation already published in Ref. [8].

Closely looking at the standard RG procedure, we notice that (i) only resonant singular terms have to be considered [15], and that (ii) the highest order Z_n has only to be computed for linear terms in τ.

With these observations, the procedure of the standard RG may be somewhat simplified, but still we need almost explicit secular terms. Hence the RG method, although it works and is often simpler than many standard techniques, is still an inelegant procedure.

The above calculation suggests, as emphasized in Ref. [8], that we do not need any insight as to the solution we wish to compute and, consequently, very mechanical procedures suffice. However, even for linear ODE's, this is not quite true, if we do not convert it into the standard form to which the WKB method is applicable [i.e., $d^2u/ds^2 = Q(s)u$]. This sort of difficulty is pointed out in Ref. [17] with a seemingly benign

$$\frac{d^2y}{dt^2} + \epsilon\frac{dy}{dt} + y = 0. \tag{2.20}$$

This equation needs some care about the initial conditions of approximate solutions, if we wish to obtain the RG equation to higher order. Thus we need some insight about the solution. Furthermore, the following counterexample due to Ei and Kuwamura is quoted also in Ref. [17]:

$$\frac{dy}{dt} = -\epsilon^2 y + \epsilon y^2 \tag{2.21}$$

cannot easily be solved by the naive subtraction scheme with the aid of the perturbative expansion in powers of ϵ.

All these difficulties are, according to our current view, due to the fact that we construct solutions and then derive coarse-grained equations. The step to construct perturbative solutions becomes, so to speak, a bottleneck between the original equation and the final slow motion equation. The only way to overcome the difficulties pointed out in Ref. [17] seems to maximally avoid explicit calculations. This becomes especially important for PDE due to the reason explained in Sec. VI. The chief aim of this paper is to provide a RG means that is maximally abstract *in order to* make it more practical and convenient than it is now.

III. PROTO-RENORMALIZATION-GROUP EQUATIONS—INTRODUCTION

In order to devise a much simpler method to obtain the renormalization-group equation, we wish to make the standard procedure in the preceding section as abstract (implicit) as possible. [However, this section may be read largely without referring to the preceding section except for the perturbation Eqs. (2.4) and (2.5) and the definition of Z_i in Eq. (2.11).]

Due to the structure of Eq. (2.4) its solution has the following structure:

$$y_1 = P_1e^{it} + Q_1e^{3it} + \text{c.c.} \tag{3.1}$$

By inspection we see that $P_1 = P_1(t,A)$ is a first degree polynomial in t, and $Q_1 = Q_1(t,A)$ is a constant. Here, their dependence on A is explicitly denoted. As seen from Eq. (2.4) they obey the following equations:

$$L_tP_1 = iA(1-|A|^2), \tag{3.2}$$

$$R_tQ_1 = \frac{1}{3}iA^3, \tag{3.3}$$

where

$$L_t \equiv \frac{d^2}{dt^2} + 2i\frac{d}{dt}, \tag{3.4}$$

$$R_t \equiv \frac{d^2}{dt^2} + 6i\frac{d}{dt} - 8. \tag{3.5}$$

The renormalized perturbation result can be written as

$$y(t) = A_R(\tau)e^{it} + \epsilon\{P_1[t, A_R(\tau)] - \hat{P}_1[\tau, A_R(\tau)]\}e^{it} + \cdots. \tag{3.6}$$

Here, $\hat{P}_1$ is the secular part of P_1 (actually $P_1 = \hat{P}_1$, if P_1 does not have the constant term; this is the usual choice of P_1 as well as ours). Since we wish to go to higher order calculations later, A's in P_1 are replaced with A_R. However, to this lowest nontrivial order, we may identify A_R with A. Also we ignore the nonsecular terms. Let L_τ be the same operator as L_t with t being replaced by τ. Then, with the aid of Eq. (3.2),

$$0 = L_\tau y = [L_\tau A_R - L_\tau \hat{P}_1(\tau, A_R)]e^{it}, \tag{3.7}$$

or

$$\left(\frac{d^2}{dt^2} + 2i\frac{d}{dt}\right)A_R(\tau) = \epsilon i A_R(1 - |A_R|^2). \tag{3.8}$$

Inspecting this equation, we realize that differentiation with respect to τ raises the power of ϵ. Therefore, to order ϵ, we may discard the second order derivative: changing the variable from τ to t, we obtain the renormalization-group equation to order ϵ:

$$\frac{dA_R}{dt} = \frac{1}{2}\epsilon A_R(1 - |A_R|^2). \tag{3.9}$$

For this reason, we call Eq. (3.8) a *proto-RG equation*; roughly speaking, a proto-RG equation is an equation that can be obtained by applying the ''simplest'' operator to the renormalized perturbation series to eliminate the general form of the subtraction terms in the series. As we see here, from this equation we can easily obtain the RG equation algebraically. The most important observation is that to the lowest nontrivial order, we do not need any explicit result. As we will see later this is a useful property for reducing PDE's.

Although we say that in our RG approach we can obtain the equation without any prior knowledge about the system or its solution, we have used implicitly that A_R is of order 1. Therefore the reader might claim that we need some prior knowledge or requirement to derive the RG equation from the corresponding proto-RG equation. When we reduce a system, we must specify what sort of solution we wish to study. If we are interested in (or expect) a small solution of order ϵ, we should obtain a different equation (i.e., a different RG equation) from the proto-RG equation that allows such a solution. If we could obtain such an RG equation, we may conclude that indeed such a global solution is allowed by the original system. If we fail, it implies that there is no consistent long time behavior for which the solution behaves as expected. Thus the criticism above is not really a criticism, but indicates the freedom of choice in reduction. Its choice is not really a concern of a reduction scheme.

Thus the strategy to compute the order ϵ singular perturbation result is to write down the proto-RG equation. This equation simply gives the RG Eq. (3.9), and the renormalized perturbation result

$$y(t) = A_R(t)e^{it} + \epsilon Q_1 e^{3it}, \tag{3.10}$$

where Eq. (3.3) or $-8Q_1 = iA_R^3/3$ gives all we want, because we know Q_1 is a constant.

Instead of calculating $\partial y/\partial\tau = 0$, calculating $L_\tau y = 0$ is practically the proto-RG method. The basic idea is that we should use the information of the system under study captured in the secular term instead of using a generic method to remove the secular terms (actually, differentiation with respect to τ does not always work). Can we use the technique to obtain higher order results in a similar fashion without any explicit calculation? As we will see in the next section the answer is yes to all orders for linear ODE's. For nonlinear ODE's, we need slightly more information than the functional forms of the solutions. Still, as is illustrated below, the calculation is far easier than most singular perturbation methods and than our previous RG approach that is already simpler than most singular perturbation methods. Equation (2.5) reads

$$\begin{aligned}\left(\frac{d^2}{dt^2} + 1\right)y_2 = {} & \{(1 - 2|A|^2)(iP_1 + \dot{P}_1) + (3iQ_1 + \dot{Q}_1)A^{*2} \\ & + (-iP_1^* + \dot{P}_1^*)A^2\}e^{it} + \{(iP_1 + \dot{P}_1)A^2 \\ & + 3i(1 - 2|A|^2)Q_1\}e^{3it} \\ & + (3iQ_1 + \dot{Q}_1)A^2 e^{5it} + \text{c.c.}\end{aligned} \tag{3.11}$$

Therefore its solution has the following form:

$$y_2 = P_2(t, A)e^{it} + Q_2(t, A)e^{3it} + S_2(t, A)e^{5it} + \text{c.c.} \tag{3.12}$$

The term containing P_2 is a resonant secular term, because it is a polynomial times e^{it} that is in resonance with the differential operator on the left hand side of Eq. (3.11). Since P_1 is degree one, $P_2(\tau)$ must be a second degree polynomial. The term containing Q_2 is a nonresonant secular term, because it is secular but does not contain the resonant factor $e^{\pm it}$. The secular nature of Q_2 is due to the appearance of P_1 together with e^{3it} in the right hand side of Eq. (3.11). S_2 is not singular (is a constant). In any case, in order to note these facts, we need only inspections. Putting Eq. (3.12) into Eq. (3.11), we see

$$\begin{aligned}L_t P_2 = {} & (1 - 2|A|^2)(iP_1 + \dot{P}_1) + (3iQ_1 + \dot{Q}_1)A^{*2} \\ & + (-iP_1^* + \dot{P}_1^*)A^2.\end{aligned} \tag{3.13}$$

So far no explicit solution has been required at all.

Our main assertion in this section is the following. The general form of the proto-RG equation (for weakly nonlinear oscillators) reads

$$L_\tau A_R = \epsilon L_\tau \hat{P}_1(\tau, A_R) + \epsilon^2 L_\tau \hat{P}_2(\tau, A_R) + \cdots, \quad (3.14)$$

where on the right hand side L_τ does not act on A_R (i.e., A_R is treated as a constant when $\hat{P}_i(\tau, A_R)$ are differentiated with respect to τ), and all the explicitly τ dependent terms must be ignored from the result (by setting $\tau = 0$ after differentiation). As we will see, the explicit expression of this formula can be obtained almost without any actual calculation.

Let us demonstrate Eq. (3.14) (including the prescription described below it). Our starting point is a general renormalization procedure, although the result we need is very simple. If we keep only the resonant secular terms [18], the naive perturbation result reads

$$y(t) = Ae^{it} + \epsilon P_1(t,A)e^{it} + \epsilon^2 P_2(t,A)e^{it} + \epsilon^3 P_3(t,A)e^{it} + \cdots + \text{c.c.} \quad (3.15)$$

Let us introduce the renormalized A as Eq. (2.11). Then, Eq. (3.15) can be written as

$$\begin{aligned} y(t) = {} & A_R(1 + \epsilon Z_1 + \epsilon^2 Z_2 + \cdots)e^{it} + \epsilon P_1(t,A_R)e^{it} \\ & + \epsilon^2 [(P_2(t,A_R) + P_1'(t,A_R)A_R Z_1)]e^{it} \\ & + \epsilon^3 \Big(P_3(t,A_R) + P_2'(t,A_R)A_R Z_1 \\ & + P_1'(t,A_R)A_R Z_2 + \frac{1}{2} P_1''(t,A_R)(A_R Z_1)^2 \Big) e^{it} + \text{c.c.}, \end{aligned} \quad (3.16)$$

where partial differentiation with respect to A_R is denoted by $'$. The renormalization constant is determined order by order as

$$\begin{aligned} & A_R Z_1 + P_1(\tau, A_R) = 0, \\ & A_R Z_2 + P_2(\tau, A_R) + P_1'(\tau, A_R)A_R Z_1 = 0, \\ & A_R Z_3 + P_3(\tau, A_R) + P_2'(\tau, A_R)A_R Z_1 + P_1'(\tau, A_R)A_R Z_2 \\ & \quad + P_1'(t, A_R)A_R Z_2 + \frac{1}{2} P_1''(\tau, A_R)(A_R Z_1)^2 = 0. \end{aligned} \quad (3.17)$$

From this, incidentally, we see that the procedure can be continued order by order indefinitely. Putting these into Eq. (3.16), we obtain the renormalized perturbation series

$$\begin{aligned} y(t) = {} & A_R e^{it} + \epsilon [P_1(t,A_R) - P_1(\tau,A_R)]e^{it} + \epsilon^2 \{P_2(t,A_R) \\ & - P_2(\tau,A_R) + [P_1'(t,A_R) - P_1'(\tau,A_R)]A_R Z_1\} e^{it} \\ & + \epsilon^3 \Big(P_3(t,A_R) - P_3(\tau,A_R) + [P_2'(t,A_R) \\ & - P_2'(\tau,A_R)]A_R Z_1 + [P_1'(t,A_R) - P_1'(\tau,A_R)]A_R Z_2 \\ & + \frac{1}{2}[P_1''(t,A_R) - P_1''(\tau,A_R)](A_R Z_1)^2 \Big) e^{it} + \cdots. \end{aligned} \quad (3.18)$$

Notice that if we put $t = \tau$, then this reduces to $y(t) = A_R(t)e^{it}$ as should be, because we have not written nonresonant terms explicitly.

Notice that Eq. (3.17) is the order by order expression of

$$A - A_R(\tau) + P(\tau, A) = 0, \quad (3.19)$$

and Eq. (3.18) is the ϵ expansion result of

$$y(t) = A_R(\tau)e^{it} + [P(t,A) - P(\tau,A)]e^{it} + \text{c.c.} \quad (3.20)$$

as expected from consistency. Applying L_τ to this equation, we obtain

$$L_\tau A_R(\tau) = L_\tau P(\tau, A). \quad (3.21)$$

After calculating the right hand side, we replace A with $A_R(\tau)Z$. Intuitively speaking, the amplitude equation for an autonomous equation should be autonomous, so the right hand side of this equation should not depend on τ explicitly. We can simply use this fact and set $\tau = 0$ in the right hand side. $Z = 1$ if we set $\tau = 0$. Thus we have shown Eq. (3.14) (including the prescription described there).

Let us demonstrate that for autonomous problems the proto-RG equation (and consequently the RG equation) must be autonomous. That is, $L_\tau A_R$ does not depend on τ explicitly. Physically, this is natural, because the long term behavior of an autonomous equation should also be autonomous. After renormalization, the perturbative result may be written order by order as

$$y(t) = A_R(t)e^{it} + F(t) + \text{c.c.}, \quad (3.22)$$

where $F(t)$ contains higher frequency terms [if we regard A_R to be constant, i.e., if we ignore the very low frequency behavior = secular behavior of $A_R(t)$, then $F(t)$ does not contain any lower frequency terms than $e^{\pm 2it}$]. If we introduce this equation into the original equation, we get

$$[L_t A_R(t)]e^{it} + \tilde{F}(t) = \epsilon N(y(t)), \quad (3.23)$$

where N is the nonlinear term in the original equation, and $\tilde{F} = L_t F$. Since the original equation is autonomous, even if we shift $t \to t + \tau$, this equation continues to hold. Note that the right hand side does not depend on time explicitly. Hence the same must be true for the left hand side. Now, we shift time as $t \to t + 2\pi$. The functional forms of $L_t A_R(t)e^{it}$ and $\tilde{F}(t)$ are order by order invariant under this shift as seen from their way of construction. Therefore $L_t A_R(t)$, which contains lower frequency behaviors, must not depend on t explicitly.

Now, let us return to a practical (or our recommended) second order calculation for the Rayleigh equation. We have

already obtained $L_\tau P_1$ as Eq. (3.2). To calculate the second order term we use Eq. (3.13) with t being replaced by τ:

$$L_\tau P_2(\tau,A_R)=(1-2|A|^2)(iP_1+\dot{P}_1)+(3iQ_1+\dot{Q}_1)A^{*2} +(-iP_1^*+\dot{P}_1^*)A^2. \tag{3.24}$$

Setting $A=A_R$ and ignoring the explicitly τ-dependent terms on the right hand side of Eq. (3.24), we can obtain the corresponding term in Eq. (3.14). Since we know that P_1 is of first degree [$P_1(0)=0$], and that Q_1 is a constant, we obtain

$$L_\tau P_2(\tau,A_R)=(1-2|A_R|^2)\dot{P}_1+3iQ_1A_R^{*2}+\dot{P}_1^*A_R^2. \tag{3.25}$$

We need

$$\dot{P}_1=\frac{1}{2}A(1-|A|^2), \quad Q_1=-\frac{i}{24}A^3, \tag{3.26}$$

which can be read off from Eqs. (3.2) and (3.3) immediately. With this minimal explicit result, we can write down the proto-RG equation to order ϵ^2 as

$$\left(\frac{d^2}{dt^2}+2i\frac{d}{dt}\right)A_R=i\epsilon A_R(1-|A_R|^2) +\epsilon^2\left(\frac{1}{8}|A_R|^4A_R+\frac{1}{2}A_R(1-|A_R|^2)^2\right). \tag{3.27}$$

In this case we can solve this equation order by order for dA_R/dt. Since

$$\frac{dA_R}{dt}=\frac{\epsilon}{2}A_R(1-|A_R|^2) \tag{3.28}$$

[recall that this was obtained above from the lowest order proto-RG equation without any explicit knowledge of P_1 and Q_1; this can of course be read off from Eq. (3.27)] to order ϵ,

$$\frac{d^2A_R}{dt^2}=\frac{\epsilon^2}{4}A_R(1-|A_R|^2)(1-3|A_R|^2)+O(\epsilon^3). \tag{3.29}$$

From Eq. (3.27) and this, we can obtain the RG equation to order ϵ^2. Needless to say, to go to the next order we need slightly more explicit results (for P_2, etc.). Still, this is much easier than the usual methods requiring detailed explicit results.

The proto-RG approach to the resonance problem may be summarized as follows:

(1) Solve the zeroth order equation, and set up perturbation equations.

(2) Write down the general form of the corrections [as, e.g., Eq. (3.12)].

(3) Find the equations for resonant secular terms [as, e.g., Eq. (3.25)].

(4) Construct the proto-RG Eq. (3.14).

(5) Reduce the proto-RG equation to the RG equation.

In this paper, we emphasize the mechanical nature of our procedure, so we adhere as much as possible to these steps [step (2) may be implicit, or may be merged into step (3)]. An important observation is that to the lowest nontrivial order, that is often the order we need, no explicit solution is required (as in the standard reductive perturbation).

To illustrate the proto-RG approach with a slightly different example, let us study the van der Pol equation:

$$\frac{d^2y}{dt^2}+y=\epsilon(1-y^2)\frac{dy}{dt}. \tag{3.30}$$

We expand as $y=y_0+\epsilon y_1+\epsilon^2y_2+\ldots$. Step (1) gives just Eq. (2.6). By inspection we may assume y_1 and y_2 have the same form as the Rayleigh equation [step (2)]. The perturbation equations read

$$\left(\frac{d^2}{dt^2}+1\right)y_1=(1-y_0^2)\frac{dy_0}{dt}, \tag{3.31}$$

$$\left(\frac{d^2}{dt^2}+1\right)y_2=(1-y_0^2)\frac{dy_1}{dt}-2y_0y_1\frac{dy_0}{dt}. \tag{3.32}$$

They give

$$L_tP_1=iA(1-|A|^2), \tag{3.33}$$

$$R_tQ_1=-iA^3, \tag{3.34}$$

and

$$L_tP_2=(1-2|A|^2)(\dot{P}_1+iP_1) -iP_1^*A^2-A^{*2}\dot{Q}_1-2iA^2P_1^*+2iA^{*2}Q_1. \tag{3.35}$$

Since we know P_1 is proportional to t and Q_1 is a constant, we need only the following formula (discarding explicitly t-dependent terms):

$$L_tP_2=(1-2|A|^2)\dot{P}_1+2iA^{*2}Q_1. \tag{3.36}$$

This is the end of step (3). Let us define P_1 and P_2 without the constant terms. The proto-RG equation has the form of Eq. (3.14). To order ϵ, it reads

$$L_tA_R=\epsilon L_tP_1, \tag{3.37}$$

that is,

$$\left(\frac{d^2}{dt^2}+2i\frac{d}{dt}\right)A_R=i\epsilon A_R(1-|A_R|^2). \tag{3.38}$$

Therefore the lowest order RG equation is given by

$$\frac{dA_R}{dt}=\frac{1}{2}\epsilon A_R(1-|A_R|^2). \tag{3.39}$$

The proto-RG equation to order ϵ^2 reads, according to Eq. (3.14),

$$\left(\frac{d^2}{dt^2}+2i\frac{d}{dt}\right)A_R=i\epsilon A_R(1-|A_R|^2) + \epsilon^2[(1-2|A|^2)\dot{P}_1+2iA^{*2}Q_1]. \tag{3.40}$$

The remaining task is to read off P_1 and Q_1 from Eqs. (3.33) and (3.34). From the former, we read off

$$\dot{P}_1=\frac{1}{2}A(1-|A|^2). \tag{3.41}$$

From Eq. (3.34), we read $Q_1=iA_R^3/8$ off. This is the end of step (4). To obtain the second order result, we need d^2A_R/dt^2 to order ϵ^2, but this can be obtained by differentiating the first order result as we did above, and the result is identical to Eq. (3.29). Hence combining these results we obtain

$$\frac{dA_R}{dt}=\frac{1}{2}\epsilon A_R(1-|A_R|^2)-\epsilon^2\frac{i}{8}A_R(1-2|A_R|^2). \tag{3.42}$$

Let us conclude this section with a further abstraction. In the above we assume a certain form of secular terms such as Eq. (3.12). Here, we do not even assume this. Let us return to the Rayleigh equation, and write its perturbation result as [step (2)]

$$y=Ae^{it}+\epsilon p_1(t)+\cdots+\text{c.c.} \tag{3.43}$$

Here, notice that in contrast to the expression such as Eq. (3.1) the structure of the first order perturbation term is not explicit as $p_1=P_1e^{it}+Q_1e^{3it}$. Let us introduce τ to denote t appearing in the secular terms and rewrite $p_1(t)$ as $\hat{p}_1(\tau,t)$. That is, $\hat{p}_1$ is obtained from p_1 by replacing t in the secular term prefactors with τ and discarding the terms that do not vanish when τ is set to zero [the so-called minimal subtraction scheme; this procedure gives $\hat{p}_1=P_1(\tau)e^{it}$ in the above example, but we do not need such an explicit form]. Then, the renormalized perturbation series reads

$$y=A_R(\tau)e^{it}+\epsilon[p_1(t)-\hat{p}_1(\tau,t)]+\cdots+\text{c.c.} \tag{3.44}$$

Introduction of τ corresponds to the splitting of d/dt to $d/dt+d/d\tau$ in d^2/dt^2+1. If we apply $(d/dt+d/d\tau)^2+1$ to $\hat{p}_1$ and factor out e^{it} from the outcome, then it must be identical to the coefficient of e^{it} on the right hand side of the first order perturbation equation governing p_1. This suggests that it is convenient to introduce the following operator:

$$\mathcal{S}\equiv\mathcal{P}_{t=\tau}\left(\frac{d^2}{d\tau^2}+2\frac{d}{d\tau}\frac{d}{dt}\right), \tag{3.45}$$

where $\mathcal{P}$ is an appropriate projection operator onto the null space (i.e., the operator factoring out the coefficient of e^{it} in the present case). (The subscript $t=\tau$ implies the prescription: t is replaced by τ after projection. This procedure is not needed to the lowest order considered here, but if we wish to use the operator in higher order calculations we must specify how to treat t on the right hand side.) Then,

$$\mathcal{S}\hat{p}_1(\tau,t)=iA(1-|A|^2). \tag{3.46}$$

This is step (3). We can use this relation to derive the proto-RG equation from Eq. (3.44) as follows. Applying $\mathcal{S}$ to Eq. (3.44), we just obtain [step (4)]

$$0=\left[\frac{\partial^2}{\partial\tau^2}+2i\frac{\partial}{\partial\tau}\right]A_R(\tau)-iA_R(1-|A_R|^2). \tag{3.47}$$

In this slightly more abstract version, to construct the operator $\mathcal{S}$ (that could appropriately be called the proto-RG operator) is the essence. How to construct it practically will be found at the end of Sec. VII. A related mathematical discussion can be found in Sec. VI.

IV. LINEAR ODE—PROTO-RG EQUATION APPROACH

When we wish to reduce a system, or to know the qualitative behavior, we need only the lowest nontrivial order result or at most the next. Therefore the proto-RG procedure is a very powerful method. When the problem is linear, we can apply the method easily to all orders. To illustrate the point, let us consider the simplest case:

$$\left(\epsilon\frac{d^2}{dt^2}+\frac{d}{dt}+1\right)y=0. \tag{4.1}$$

If we expand

$$y=y_0+\epsilon y_1+\cdots+\epsilon^n y_n+\cdots, \tag{4.2}$$

then order by order we obtain ($n=0,1,2,\ldots$, $y_{-1}\equiv 0$)

$$\frac{dy_n}{dt}+y_n=-\frac{d^2y_{n-1}}{dt^2}. \tag{4.3}$$

The unperturbed result reads $y_0=Ae^{-t}$, and the solution to Eq. (4.3) can be written ($n=1,2,\ldots$) as $y_n=AP_ne^{-t}$, where P_n are polynomials determined recursively as

$$\frac{dP_n}{dt}=-\frac{d^2P_{n-1}}{dt^2}+2\frac{dP_{n-1}}{dt}-P_{n-1} \tag{4.4}$$

with the initial condition $P_n(0)=0$. This condition allows us to identify the secular term $\hat{P}_n$ and P_n itself.

Using these polynomials, the naive perturbation result reads

$$y(t)=A[1+\epsilon P_1(t)+\cdots+\epsilon^nP_n(t)+\cdots]e^{-t}. \tag{4.5}$$

Thanks to the linearity of the problem P_n are independent of A. The secular part of P_n is P_n itself because of the initial condition. The renormalization of A can be written as $A = ZA_R(\tau)$. Introducing this into Eq. (4.5), we obtain

$$y(t) = A_R(\tau) Z[1 + \epsilon P_1(t) + \cdots + \epsilon^n P_n(t) + \cdots] e^{-t}. \tag{4.6}$$

If we choose $t - \tau$ to be higher order than any power of ϵ, then obviously

$$Z^{-1} = 1 + \epsilon P_1(\tau) + \cdots + \epsilon^n P_n(\tau) + \cdots . \tag{4.7}$$

That is,

$$A_R = A[1 + \epsilon P_1(\tau) + \cdots + \epsilon^n P_n(\tau) + \cdots]. \tag{4.8}$$

In other words, summing all the divergent terms may be understood as renormalization as in the naive renormalization we encounter in, e.g., many-body theory. This is a general feature of linear problems.

From Eq. (4.4) through an order by order calculation, we obtain

$$\frac{dA_R}{d\tau} = \epsilon \left(-\frac{d^2 A_R}{d\tau^2} + 2\frac{dA_R}{d\tau} - A_R \right). \tag{4.9}$$

Conversely, if we solve this equation recursively in powers of ϵ with the zeroth order result being constant, we can determine P_n order by order. We see that Eq. (4.9) is obtained simply by substituting $y = A_R(t) e^{-t}$ into the original equation. That is, Eq. (4.9) is the proto-RG equation, and from this we know trivially that the perturbative RG result is correct to all orders.

The RG equation can be obtained by solving Eq. (4.9) for $dA_R/d\tau$ order by order as follows. Noticing that differentiation is equivalent to raising the power in ϵ, we obtain to the lowest order

$$\frac{dA_R}{d\tau} = -\epsilon A_R . \tag{4.10}$$

Using this to the rhs, we get to order ϵ^2

$$\frac{dA_R}{d\tau} = -(\epsilon + 2\epsilon^2) A_R . \tag{4.11}$$

Differentiating this further gives

$$\frac{d^2 A_R}{d\tau^2} = -\epsilon \frac{dA_R}{d\tau} = \epsilon^2 A_R \tag{4.12}$$

to order ϵ^2. From this and Eq. (4.11) we can obtain, for example, to order ϵ^3

$$\frac{dA_R}{d\tau} = -(\epsilon + 2\epsilon^2 + 5\epsilon^3) A_R , \tag{4.13}$$

etc.

One might say that the method is not so useful for linear problems, since, after all, the problems are simple; the same results can be obtained easily by, e.g., the Lie group method [16]. This is very often the case. However, if the Lie group is not acting on a simple space ($\boldsymbol{R}$ or $\boldsymbol{C}$), the proto-RG approach seems sometimes simpler. Let us demonstrate the point with simple examples.

Let us first consider

$$\frac{d^2 y}{dt^2} + (2 + \epsilon)\frac{dy}{dt} + y = 0. \tag{4.14}$$

The decay mode of the solution depends on the sign of ϵ, and $\epsilon = 0$ may be understood as a bifurcation point. The unperturbed solution has the form $(a + bt)e^{-t}$, where a and b are constants. Therefore let us set $y = A(t)e^{-t}$, and write down the equation for A. This is the proto-RG equation in this case:

$$\frac{d^2 A}{dt^2} = \epsilon \left(A - \frac{dA}{dt} \right). \tag{4.15}$$

From this the lowest order RG equation reads (although we must admit that we need no such fancy name for these equations)

$$\frac{d^2 A}{dt^2} = \epsilon A. \tag{4.16}$$

In this case Eq. (4.15) is actually the RG equation (to all orders) and is not the first order equation. We would need some insight to derive this from naive perturbation results. At least in our original naive RG, the problem is not very easy.

An interesting point of this example is that the RG equation *of* this RG equation is useful to study its long-time behavior. Scaling the time variable as $t = \sqrt{|\epsilon|}\, s$, the RG equation becomes

$$\frac{d^2 A}{ds^2} = \mathrm{sgn}(\epsilon) \left(A - \sqrt{|\epsilon|}\, \frac{dA}{ds} \right). \tag{4.17}$$

Consider the $\epsilon < 0$ case. Its unperturbed solution reads $A = Be^{is} + \text{c.c.}$ The proto-RG equation can be obtained by assuming that B is time dependent:

$$\frac{d^2 B}{ds^2} + 2i\frac{dB}{ds} = \sqrt{-\epsilon} \left(iB + \frac{dB}{ds} \right). \tag{4.18}$$

Hence the lowest order RG equation is (B is assumed to be not infinitesimally small)

$$\frac{dB}{ds} = -\frac{\sqrt{-\epsilon}}{2} B. \tag{4.19}$$

This is a convenient occasion to consider the counter or difficult examples mentioned in Sec. II. For Eq. (2.20) the proto-RG equation reads

$$\frac{d^2A}{dt^2}+2i\frac{dA}{dt}=-\epsilon\left(iA+\frac{dA}{dt}\right). \tag{4.20}$$

If we may assume that A is of order 1, then we can iteratively solve this for dA/dt. To the lowest order we obtain

$$\frac{dA}{dt}=-\frac{\epsilon}{2}A. \tag{4.21}$$

From this we obtain

$$\frac{d^2A}{dt^2}=\frac{\epsilon^2}{4}A \tag{4.22}$$

to order ϵ^2. Hence to order ϵ^2 the RG equation reads

$$\frac{dA}{dt}=-\left(\frac{\epsilon}{2}+\frac{\epsilon^2}{8}i\right)A. \tag{4.23}$$

For Eq. (2.21), the proto-RG equation is the original equation itself:

$$\frac{dA}{dt}=\epsilon A^2-\epsilon^2 A. \tag{4.24}$$

The problem is that the A we are interested in is of order ϵ or smaller. Therefore both terms on the right hand side can be comparable, so no further reduction is possible. That is, we must interpret that the proto-RG equation is the RG equation itself. The lesson of this example is that, although we tend to claim in our RG approach that we do not need any *a priori* estimate of the solution, we must know at least what solution (around what fixed point, for example) we wish to study and its rough order.

As a not-so-trivial example of reducing the proto-RG to the RG equation, let us consider the bifurcation problem of the Mathieu equation: the problem is to find the range of ω such that

$$\frac{d^2y}{dt^2}+y=-\epsilon[\omega+2\cos(2t)]y \tag{4.25}$$

does not have a bounded solution. Although this is not an autonomous equation, for linear problems, it is easy to see that the proto-RG method works to all orders. The unperturbed solution reads

$$y_0=Ae^{it}+A^*e^{-it}. \tag{4.26}$$

We must first write down the proto-RG equation. Assuming that A is a function of time, and introducing Eq. (4.26) into Eq. (4.25), we get

$$\frac{d^2A}{dt^2}+2i\frac{dA}{dt}+\text{c.c.}=-\epsilon(\omega A+A^*)-\epsilon Ae^{2it}+\text{c.c.} \tag{4.27}$$

Here, c.c. denotes the complex conjugate terms. Although we may call this the proto-RG equation of the Mathieu equation, it is not very illuminating. One might guess that, since the original equation is real, we have only to consider the following equation

$$\frac{d^2A}{dt^2}+2i\frac{dA}{dt}=-\epsilon(\omega A+A^*)-\epsilon Ae^{2it}. \tag{4.28}$$

However, this is wrong, because the nonautonomous driving term mixes A^* and A.

The easiest method that still allows us to avoid explicit calculation of perturbative results is to expand as

$$y=A(t)e^{it}+\epsilon B(t)e^{3it}+\epsilon^2C(t)e^{5it}+\cdots+\text{c.c.} \tag{4.29}$$

This form is easily guessed from the fact that $e^{\pm 2it}$ appears with ϵ. The procedure is to get the equations (proto-RG equations) for the coefficients, and then reduce them to the equation of A alone. To order ϵ^2 we have

$$\frac{d^2A}{dt^2}+2i\frac{dA}{dt}=-\epsilon(\omega A+A^*)-\epsilon^2B. \tag{4.30}$$

The equation for B is

$$\epsilon\left(\frac{d^2B}{dt^2}+6i\frac{dB}{dt}-8B\right)=-\epsilon A-\epsilon^2\omega B+\epsilon^3C. \tag{4.31}$$

Since derivatives give higher order powers of ϵ, we see from this $B=A/8$ to order ϵ. Hence to order ϵ^2 the proto-RG equation is reduced to

$$\frac{d^2A}{dt^2}+2i\frac{dA}{dt}=-\epsilon(\omega A+A^*)-\epsilon^2\frac{A}{8}. \tag{4.32}$$

It is easy to reduce this to a first order differential equation from which we can immediately read off the stability requirement. Recall that the ordinary singular perturbation requires expansion in powers of $\epsilon^{1/2}$.

Another simple example of the nonautonomous equation that requires some care in reducing the proto-RG to the RG equation is

$$\frac{d^2u}{dt^2}+u=\epsilon tu. \tag{4.33}$$

This is linear, but we use the order ϵ term as a perturbation. We use the same zeroth order result as before: $u_0=Ae^{it}+\text{c.c.}$ Then, the first order correction u_1 obeys

$$\frac{d^2u_1}{dt^2}+u_1=tAe^{it}+\text{c.c.}, \tag{4.34}$$

u_1 has the following form:

$$u_1=P_1e^{it}+\text{c.c.}, \tag{4.35}$$

where with the aid of L in Eq. (3.4)

$$LP_1 = \epsilon t A. \tag{4.36}$$

Therefore the proto-RG equation is

$$\left(\frac{d^2}{dt^2} + 2i\frac{d}{dt}\right)A = \epsilon t A. \tag{4.37}$$

To reduce this equation to a first order equation, we must be slightly careful, because dA/dt and d^2A/dt^2 are both of order ϵ. This can be seen as follows. Obviously from Eq. (4.37) $dA/dt = O[\epsilon]$. Differentiating Eq. (4.37) once, we obtain

$$\left(\frac{d^3}{dt^3} + 2i\frac{d^2}{dt^2}\right)A = \epsilon\left(A + t\frac{dA}{dt}\right). \tag{4.38}$$

The right hand side is still of order ϵ, so $d^2A/dt^2 = O[\epsilon]$. Differentiating Eq. (4.38) once more, we obtain

$$\left(\frac{d^4}{dt^4} + 2i\frac{d^3}{dt^3}\right)A = \epsilon\left(2\frac{dA}{dt} + t\frac{d^2A}{dt^2}\right). \tag{4.39}$$

Therefore we conclude that d^3A/dt^3 or higher order derivatives are higher order than ϵ. Hence from Eq. (4.38) we obtain

$$\frac{d^2A}{dt^2} = -\frac{i}{2}\epsilon A. \tag{4.40}$$

Using this in Eq. (4.37), we obtain to order ϵ

$$\frac{dA}{dt} = -\frac{i}{2}t\epsilon A + \frac{\epsilon}{4}A. \tag{4.41}$$

This is indeed the correct RG equation to order ϵ obtained by a more explicit conventional procedure explained in Ref. [8]. The reader may say that the above calculation is possible because the time-dependent factor (t in this case) is simple; if it is a general time-dependent function $f(t)$, then the procedure would not work. A short reply to this criticism is: if approaches similar to the above do not work, then RG procedures using explicit singular terms are hopelessly complicated (this means the conventional singular perturbations are hopeless as well). If some higher order derivatives vanish identically (i.e., if f is a polynomial), the procedure works. Otherwise, there is no mechanical way. That is, there is no mechanical way to obtain the global behavior.

V. BEYOND ALL ORDERS

As we have seen in the preceding section, our RG method works to all orders. However, it is clear that the method explained there cannot give the other solution of Eq. (4.1) whose leading order behavior is $e^{-t/\epsilon}$. To obtain such a term is called the problem beyond all orders. As we have seen above, the RG approach is just to sum the secular terms for linear problems, so that we discuss only the summation in this section and will not explicitly mention RG.

Let us reconsider Eq. (4.1) with the aid of the Laplace transformation. Let us write the Laplace transform of y_n to be Y_n. Then, formally Eq. (4.3) reads

$$(s+1)Y_n = -s^2 Y_{n-1}. \tag{5.1}$$

It is easy to solve this iteratively as

$$Y_n = \left(\frac{-s^2}{1+s}\right)^n Y_0. \tag{5.2}$$

If we sum all these terms, we get

$$Y = Y_0\frac{1}{1+\epsilon s^2/(1+s)} = \frac{(1+s)Y_0}{1+s+\epsilon s^2}. \tag{5.3}$$

Thus we can recover the original differential Eq. (4.1), so that obviously we can recover the transcendental behavior $\simeq e^{-t/\epsilon}$. The reason why we could not get such a term in the preceding section was simply that we chose the initial condition such that $y_0(0) = A$ and $y_0'(0) = -A$ (as is demanded by the zeroth order equation). This condition exactly removes the contribution from the zero of the denominator of Eq. (5.3) that behaves like $1/\epsilon$. This is also the only condition we can impose consistently to the first order differential equation.

In short, the full information about the transcendental terms is still retained in the perturbative result itself, but we (so to speak, meticulously) discard it through imposing a special auxiliary condition in the usual singular perturbation approach. Therefore if we stop discarding the full information, or if we try to retain the extra information needed for the transcendental terms, we should be able to get the result even beyond all orders from perturbative results.

One (and the conventional) way to retain two auxiliary conditions is to scale the variable as $t = \epsilon s$ to magnify the boundary layer. Then, the perturbation term becomes nonsingular, and we obtained the transcendental terms as well. However, here we avoid this approach and keep the singular nature of the perturbation in the most naive way. The reader may well say that practically the conventional method is simple and standard enough, so there is no point to give a ''nonscaling'' approach. First of all, we wish to show that it is untrue that the scaling of the variable is necessary to get the result beyond all orders contrary to the general belief. The main aim of this section is to point out:

(i) The main difficulty of the naive perturbation approach is solely due to its incapability of accommodating all the auxiliary conditions for the original problem (due to the decrease of the order of the differential equation). If we can overcome this difficulty, we can recover even the ''beyond all orders'' results perturbatively without rescaling the variable.

(ii) The resultant approach has the same structure mathematically as the so-called exact WKB theory, the only rigorous singular perturbation theory beyond all orders [20].

Let us consider the simplest example with a singular behavior:

$$\epsilon \frac{dy}{dt} + y = 0. \tag{5.4}$$

Its general solution is $y = Ae^{-t/\epsilon}$. If we perform the expansion $y = y_0 + \epsilon y_1 + \ldots$, then we obtain $y = 0$, which is consistent with the asymptotic expansion of the exact solution in powers of ϵ. The problem is that the zeroth order equation is not even an ODE in this case, so that no initial condition can be imposed.

The most natural approach to rescue the situation seems to be as follows. An initial condition may be imposed with the aid of the delta function as

$$\epsilon \frac{dy}{dt} + y = \alpha \delta(t) \tag{5.5}$$

with a homogeneous initial condition $y(0) = 0$. If the initial condition for the original problem is $y(0) = A$, then $\alpha = \epsilon A$, so that if we treat the delta function as an ordinary function, then one might observe that when we drop the derivative, we should drop the delta function term as well. However, our experience with the Laplace transformation tells us that we should retain the delta function term to the zeroth order (that is, ϵ times δ or its derivatives must be treated in a special way).

The easiest way to solve Eq. (5.5) is with the aid of the Laplace transformation, but to explore the possibility of studying nonconstant coefficient equations, we avoid this approach. The zeroth order equation reads

$$y_0 = \alpha \delta(t). \tag{5.6}$$

The perturbation equations read

$$y_n = -\frac{dy_{n-1}}{dt}. \tag{5.7}$$

Hence we obtain

$$y_n = \alpha \left(-\frac{d}{dt} \right)^n \delta(t), \tag{5.8}$$

so that

$$y = \sum_{n=0}^{\infty} \alpha \left(-\epsilon \frac{d}{dt} \right)^n \delta(t). \tag{5.9}$$

To sum this highly singular series, we use the Borel summation method. Let

$$B(s) \equiv \sum_{n=0}^{\infty} \alpha \frac{1}{n!} \left(-s \frac{d}{dt} \right)^n \delta(t) = \alpha \delta(t-s). \tag{5.10}$$

Then, the Borel summation result reads

$$y = \frac{1}{\epsilon} \int_0^{\infty} B(s) e^{-s/\epsilon} ds = \frac{\alpha}{\epsilon} e^{-t/\epsilon}. \tag{5.11}$$

Thus we have obtained the result beyond all orders from a ''naive'' perturbative calculation [19].

If the equation is a second order equation, then the initial condition for $y_0(0)$ and $y_0'(0)$ may be imposed by an inhomogeneous term consisting of a linear combination of $\delta(t)$ and $\delta'(t)$ with homogeneous initial conditions on y_0.

From the above calculations, it is tempting to conjecture that perturbative calculations, appropriately organized, can always give us all the information about the original equation. Consequently, the results beyond all orders can be obtained perturbatively. A crucial ingredient is to retain the degrees of freedom (flexibility of accommodating sufficiently many auxiliary conditions) in the original problem in the perturbative procedure.

We wish to point out that the essence of the above calculation, the formal expansion + Borel transformation with respect to the expansion parameter, is the same as that of the so-called exact WKB analysis [20].

VI. GENERAL CONSIDERATION ON PARTIAL DIFFERENTIAL EQUATIONS

In the case of ODE's, the secular terms due to the perturbation seem unambiguously identifiable, and they are obtainable by the Lagrange method. The operator in the proto-RG equation (and eventually that in the RG equation) is chosen to remove these divergences. If we inspect the same procedure for PDE's, we realize that the situation is more complicated, because the solution to the inhomogeneous equation is generally not unique.

For the illustration sake, let us study a simple example,

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2} \right) \phi = 1, \tag{6.1}$$

for $t > 0$ and $z \in \boldsymbol{R}$. t or $-z^2/2$ is a special solution, and the general solution [21] to this equation may be obtained from the general solution to the homogeneous diffusion equation plus t, for example. The problem is clear: even the divergent (or secular) terms need not be unique. If we try to remove the (space-time) secular term, depending on the choice of our secular solution to the inhomogeneous equation, the resultant RG equations are different, because, according to the procedure employed in our previous papers, the RG equation is determined by the condition to remove the secular terms. One prototypical method was utilized in Ref. [8], but was far from systematic and complete. This is the reason why there were controversies [9,12] as in the case of the Swift-Hohenberg equation [26].

If Eq. (6.1) is a perturbation equation, then the key step to construct the proto-RG equation should be to find an operator that maps the class Ω of (space and/or time) secular functions that satisfy Eq. (6.1) to 1 (or something known or tractable). Again, there may be many different operators that can do the job. We wish to map the smallest set containing Ω (hopefully itself) to a known object. We have not yet codified all of the characterizations of the required map (i.e., differential operator), but at least it is clear that we must look for the lowest order operator (''antiprincipal'' part) among

them that maps Ω to, say, 1. This is in accordance with our interest in the most global space-time features of the solution.

In any case, the crucial point is that we must deal with a set of functions, not individual secular terms, if we wish to perform renormalization-group theoretical reduction. Therefore the idea of the proto-RG approach introduced in the preceding sections, or more generally the strategy to avoid explicit calculations as much as possible, becomes more relevant to PDE's than to ODE's. To illustrate our approach, we give several examples.

VII. EVOLUTION EQUATIONS

A. Interface dynamics

Consider the following simple semilinear parabolic equation (the time-dependent Ginzburg-Landau equation for the φ^4 free energy functional without conservation):

$$\frac{\partial \psi}{\partial t} = \psi - \psi^3 + \Delta \psi. \tag{7.1}$$

It has a plane kink solution ψ_K,

$$\psi_K(z) = \tanh\left(\frac{z}{\sqrt{2}}\right), \tag{7.2}$$

that describes an interface between two segregated domains. $\psi_K(z-f)$ describes a kink displaced by f in the z direction, if f is constant. Note that $z=f$ describes the interface position. If the interface is not flat, that is, if f is a nonconstant function of x and y, then $\psi_K(z-f)$ is no more a solution to Eq. (7.1). Assuming $\psi_K(z-f)$ with nonconstant f as an initial condition of Eq. (7.1), we wish to describe the evolution of the system. If the deviation of f from being constant is locally not great, then the time evolution of the system should be described by the time evolution of the interface shape, that is, in terms of the time evolution of f.

We split the rhs of Eq. (7.1) into two parts as

$$\frac{\partial \psi}{\partial t} = \left(\psi - \psi^3 + \frac{\partial^2 \psi}{\partial z^2}\right) + \Delta_2 \psi, \tag{7.3}$$

where Δ_2 is the Laplacian acting only on the variables x and y. The last term is regarded as a perturbation that becomes nonzero due to nonconstancy of f. Note that its smallness is not due to any small parameter associated with the operator, but due to the gentle dependence of the function on x and y upon which Δ_2 acts (that is, due to our choice of the initial condition). We will not write explicitly the parameter denoting the smallness of the perturbation and the perturbative terms in the solution [we could write $f(\epsilon x, \epsilon y)$ with small ϵ to be explicit]. Since we study only the lowest nontrivial order, clear indication of the order is not needed. Notice that $\psi_0 \equiv \psi_K(z-f(x,y))$ is a solution to the unperturbed equation.

Writing the deviation of the true solution from ψ_0 as φ,

$$\psi = \psi_0 + \varphi, \tag{7.4}$$

we get to, the lowest nontrivial order [step (1)],

$$\left(\frac{\partial}{\partial t} - L\right)\varphi = -H\psi_K'(z-f) + (\nabla_2 f)^2 \psi_K''(z-f), \tag{7.5}$$

where L is the linearized operator obtained from the spatial part of the unperturbed equation:

$$L = 1 - 3\psi_K^2(z-f) + \partial_z^2. \tag{7.6}$$

H is twice the mean curvature of the interface, and ∇_2 is the gradient operator acting on the functions of x and y only. When a wakelike disturbance is generated, φ does not vary slowly with respect to (x,y) as assumed in Eq. (7.5). However, the disturbance does not contribute to the leading order variation of f. Notice that $\psi_K'(z-f)$ is the zero eigenfunction of the operator L (corresponding to the Nambu-Goldstone mode). Therefore this can be the source of the secular term. Notice further that ψ_K'' is orthogonal to this function in the L_2 sense (as a function of z).

Let us renormalize Eq. (7.4) as (we follow the abstract procedure at the end of Sec. III)

$$\psi(t,\boldsymbol{r}) = \psi_K(z - f_R(\tau,\zeta,x,y)) + \varphi(t,\boldsymbol{r}) - \hat{\varphi}(\tau,\zeta,t,\boldsymbol{r}), \tag{7.7}$$

where the last subtraction describes the secular term in φ with the secular t dependence being replaced with τ and that of z by ζ (with physical insight one can say that there is no secular dependence on z to the lowest nontrivial order, but we do not use this insight); t and $\boldsymbol{r}$ may remain [and usually do as in Eqs. (3.6) or (3.44)]. This splitting of the secular behavior corresponds to replacing the differential operators as

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial z} \to \frac{\partial}{\partial z} + \frac{\partial}{\partial \zeta} \tag{7.8}$$

in the construction of the proto-RG operator. The needed projection is the projection onto the subspace spanned by ψ_K' with respect to the scalar product of the L_2 space of functions of z in the present case. Equation (7.5) implies [step (3)]

$$\mathcal{S}\hat{\varphi} = -H, \tag{7.9}$$

or

$$\left(\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \zeta^2} - 2\frac{\partial}{\partial \zeta}\frac{\partial}{\partial z}\right)\hat{\varphi} = -H\psi_K'(z-f) \tag{7.10}$$

modulo the component orthogonal to ψ_K'. Applying the proto-RG operator $\mathcal{S}$ to Eq. (7.7), we obtain

$$\left[\left(\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \zeta^2} - 2\frac{\partial}{\partial \zeta}\frac{\partial}{\partial z}\right)\psi_K(z - f_R(\tau,\zeta,x,y))\right]_{\|} = -[H(x,y,\tau)\psi_K'(z-f_R)]_{\|}, \tag{7.11}$$

where $\|$ denotes the component in the null space of L. Computing this explicitly, we arrive at [step (4)]

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial z^2}\right) f_R(t,z,x,y) = -H(x,y,t), \tag{7.12}$$

where we have used that ψ_K'' is orthogonal to ψ_K'. The equation in the orthogonal complement of Eq. (7.11) must also vanish [22]. This gives the condition that $\partial f_R/\partial\zeta =$ constant (to this order), so that actually, there is no second derivative of f_R in Eq. (7.12). Thus the outcome is equivalent to the famous Allen-Cahn [23] equation governing the normal velocity of the interface. The derivation is an RG version of Kawasaki and Ohta [24].

In the above, we have assumed that the initial deviation is explicitly given by the nonflat interface. Instead, we may assume that the initial deviation is given by

$$\psi = \psi_0 + \varphi_1 + \varphi_2 + \cdots, \tag{7.13}$$

where $\psi_0 = \psi_K(z-f_0)$ ($f_0 =$ const) is an exact solution of Eq. (7.1) and φ_j is the jth order perturbed field. This time, the first and the second order deviations obey the following linearized equations:

$$\left(\frac{\partial}{\partial t}-\Delta_2-L\right)\varphi_1 = 0, \tag{7.14}$$

$$\left(\frac{\partial}{\partial t}-\Delta_2-L\right)\varphi_2 = -3\psi_0\varphi_1^2, \tag{7.15}$$

where L is given by Eq. (7.6) with a constant $f=f_0$. Note that Eqs. (7.14) and (7.15) have secular solutions of the form $P(t,\boldsymbol{r})\psi_K'$ and $Q(t,\boldsymbol{r})\psi_K' + P(t,\boldsymbol{r})^2\psi_K''/2$, respectively, where P and Q are polynomials of t and $\boldsymbol{r}$. Here, it is easy to see that P does not depend on z and $\partial_z Q$ is constant [16]. Let us renormalize Eq. (7.13) up to the second order as

$$\psi(t,\boldsymbol{r}) = \psi_K(z-f_R(\tau,\boldsymbol{\rho})) + \varphi_1(t,\boldsymbol{r}) + \varphi_2(t,\boldsymbol{r}) - \hat{\varphi}_1(\tau,\boldsymbol{\rho},z) - \hat{\varphi}_2(\tau,\boldsymbol{\rho},z), \tag{7.16}$$

where the last subtractions remove the secular terms in φ_j with the secular $t,\boldsymbol{r}$ dependence in $P(t,\boldsymbol{r})$ being replaced with $\tau,\boldsymbol{\rho}=(\xi,\eta,\zeta)$; z in ψ_K' and ψ_K'' may remain. This splitting of the secular behavior corresponds to replacing the differential operators as Eq. (7.8) and

$$\frac{\partial}{\partial \boldsymbol{r}} \to \frac{\partial}{\partial \boldsymbol{r}} + \frac{\partial}{\partial \boldsymbol{\rho}} \tag{7.17}$$

in the construction of the proto-RG operator. The needed projection is the projection onto the subspace spanned by ψ_K' with respect to the scalar product of the L_2 space of functions of z in the present case. Equations (7.14) and (7.15) imply

$$\mathcal{S}(\hat{\varphi}_1 + \hat{\varphi}_2) = 0, \tag{7.18}$$

or,

$$\left(\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\eta^2}-2\frac{\partial}{\partial\boldsymbol{\rho}}\cdot\frac{\partial}{\partial\boldsymbol{r}}\right)\sum_{j=1}^{2}\hat{\varphi}_j = 0, \tag{7.19}$$

where ∂_ζ^2 is omitted from the proto-RG operator since secular terms in $\hat{\varphi}_j$ are, at most, polynomials of degree one with respect to ζ. Applying this modified proto-RG operator to Eq. (7.16) and noting Eq. (7.18), we obtain

$$\left[\left(\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\eta^2}-2\frac{\partial}{\partial\boldsymbol{\rho}}\cdot\frac{\partial}{\partial\boldsymbol{r}}\right)\psi_K(z-f_R(\tau,\boldsymbol{\rho}))\right]_{\|} = 0, \tag{7.20}$$

where $\|$ denotes the component in the null space of L. Using the fact that ψ_K'' is orthogonal to ψ_K' in the L_2 sense (as a function of z), we arrive at

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right) f_R(t,\boldsymbol{r}) = 0. \tag{7.21}$$

This is exactly Eq. (7.12), because $H=-(\partial_x^2+\partial_y^2)f_R$.

In the previous RG approach [25] a special form of the singular term $-tH\psi_K'(z-f(x,y))$ was used. The rest is the same but is maybe simpler than the derivation above. The procedure can be justifiable *a posteriori*, but a fair amount of insight was required.

The merit of the RG derivation of the Allen-Cahn equation may be an explicit information easily obtained from its derivation about its valid time range as discussed in Ref. [25]. The result should be meaningful after renormalization up to the time scale of $tH\sim 1$. Since $H\sim t^{-1/2}$ (as, e.g., can be seen from dimensional analysis), the Allen-Cahn equation cannot be reliable forever. Physically, this is obvious, because we have ignored the displacement of the average position of the interface in the derivation.

B. Phase equation for spatially modulated oscillation

Let $X\in\boldsymbol{R}^n$ be an n-dimensional vector (describing concentrations of n chemical species). Suppose

$$\frac{dX}{dt} = F(X), \tag{7.22}$$

where F is a vector-valued function, has a periodic solution $X_0 = X_0(\omega t+\varphi)$, where the phase φ can be any function of space determined by the initial condition. Let us consider a system that can be spatially inhomogeneous governed by the following reaction diffusion equation:

$$\frac{\partial X}{\partial t} = F(X) + D\Delta X. \tag{7.23}$$

We study the solution of this equation close to the spatially uniform X_0. Again as in the first procedure in the preceding subsection, the magnitude of the last term, the perturbation term, depends on the phase function φ. We choose this function to be only gently space dependent. Let us write

$$X = X_0 + X_1 + \cdots . \tag{7.24}$$

The first order term obeys

$$\frac{\partial X_1}{\partial t} = \nabla_X F(X_0(\omega t + \varphi)) X_1 + D\Delta X_0 , \tag{7.25}$$

where ∇_X is the gradient operator in $\boldsymbol{R}^n$. Since X_0', where $'$ denotes the differentiation with respect to the phase, is a null eigenvector of the linear operator acting on X_1 in Eq. (7.25), the solution to this equation may be renormalized as [step (2)]

$$X(t,\boldsymbol{r}) = X_0(\omega t + \varphi_R(\tau,\boldsymbol{r})) + X_1(t,\boldsymbol{r}) - \hat{X}_1(\tau,t,\boldsymbol{r}), \tag{7.26}$$

where $\hat{X}_1$ is X_1 with its secular term variables replaced as $t \to \tau$. From Eq. (7.25) it is obvious that $\mathcal{S} = P\partial/\partial\tau$, where P is the projection operator onto the null space of the linear operator $\nabla_X F(X_0)$. Therefore we have [step (3)]

$$\left[\frac{\partial}{\partial\tau}\hat{X}_1\right]_{\|} = D[\Delta X_0]_{\|} , \tag{7.27}$$

where $[\ ]_{\|}$ denotes the projection onto the null space of the linear operator $\nabla_X F(X_0)$. With the help of this Eq. (7.26) gives the following proto-RG equation [step (4)]

$$\left[\frac{\partial}{\partial\tau} X_0(\omega t + \varphi_R(\tau,\boldsymbol{r}))\right]_{\|} = D[\Delta X_0(\omega t + \varphi_R(\tau,\boldsymbol{r}))]_{\|} , \tag{7.28}$$

where we have replaced φ in the rhs with the renormalized counterpart (consistent to this order). It reads more explicitly as

$$\left[X_0' \frac{\partial\varphi_R}{\partial\tau}\right]_{\|} = D[X_0'\Delta\varphi_R + X_0''(\nabla\varphi_R)^2]_{\|} . \tag{7.29}$$

From this we obtain [step (5)]

$$\frac{\partial\varphi_R}{\partial t} = \beta\Delta\varphi_R + \alpha|\nabla\varphi_R|^2, \tag{7.30}$$

where

$$\alpha = \langle u_0 \cdot DX_0''\rangle / \langle u_0 \cdot X_0'\rangle, \tag{7.31}$$

$$\beta = \langle u_0 \cdot DX_0'\rangle / \langle u_0 \cdot X_0'\rangle. \tag{7.32}$$

Here, u_0 is the left null eigenfunction of $\nabla_X F(X_0)$ [i.e., $\langle u_0 \cdot \nabla_X F(X_0)\rangle = 0$], and $\langle\ \rangle$ is the integration over $[0,2\pi]$ with respect to the phase variable. This scalar product realizes the needed projection in the proto-RG operator. The result is the standard phase equation.

Again as in the preceding subsection, we can assume that the unperturbed X_0 is strictly spatially uniform. In this case the first and second order terms obey

$$\frac{\partial X_1}{\partial t} = \nabla_X F(X_0(\omega t + \varphi)) X_1 + D\Delta X_1 , \tag{7.33}$$

$$\frac{\partial X_2}{\partial t} = \nabla_X F(X_0) X_2 + D\Delta X_2 + (1/2)\nabla_X^2 F(X_0) : X_1^2 . \tag{7.34}$$

Since Eqs. (7.33) and (7.34) have secular solutions of the same form $P(t,\boldsymbol{r})X_0'$ and $Q(t,\boldsymbol{r})X_0' + P^2(t,\boldsymbol{r})X_0''/2$ as in the preceding subsection, where $'$ denotes the differentiation with respect to the phase, the solution to this equation may be renormalized as

$$X(t,\boldsymbol{r}) = X_0(\omega t + \varphi_R(\tau,\boldsymbol{\rho})) + X_1(t,\boldsymbol{r}) + X_2(t,\boldsymbol{r}) - \hat{X}_1(\tau,t,\boldsymbol{\rho}) - \hat{X}_2(\tau,t,\boldsymbol{\rho}), \tag{7.35}$$

where $\hat{X}_j$ is X_j with its secular term variables replaced as $t \to \tau$ and $\boldsymbol{r} \to \boldsymbol{\rho}$. From Eqs. (7.33) and (7.34), we obtain

$$\mathcal{S}\sum_{j=1}^{2} \hat{X}_j = 0, \tag{7.36}$$

where

$$\mathcal{S} = P\left[\frac{\partial}{\partial\tau} - \left(\frac{\partial^2}{\partial\boldsymbol{\rho}^2} + 2\frac{\partial}{\partial\boldsymbol{\rho}}\frac{\partial}{\partial\boldsymbol{r}}\right)D\right], \tag{7.37}$$

where P is the projection operator onto the null space of the linear operator $\partial_t - \nabla_X F(X_0)$. Then, Eq. (7.35) gives the following proto-RG equation:

$$\mathcal{S}X_0(\omega t + \varphi_R(\tau,\boldsymbol{\rho})) = 0, \tag{7.38}$$

which is identical to Eq. (7.29). The rest is the same.

In this second derivation, we have not assumed explicitly that the spatial variation of the phase φ is small. Instead, the secular solutions of Eqs. (7.33) and (7.34) are chosen so that they are eliminated by renormalizing the phase [16]. This choice of the secular solutions implies that small perturbations around the exact solution X_0 with a constant phase yield only a small deviation from the constant phase for $t \sim O(1)$ and $\boldsymbol{r} \sim O(1)$. The resultant phase Eq. (7.30) shows that the phase deviates substantially from the constant value for a longer time or a larger spatial scale.

C. Taniuti-Wei reductive perturbation

The original Taniuti-Wei reductive perturbation [2] applied to simple waves can be given RG reductively as follows. The starting equation is

$$\frac{\partial}{\partial t}U + A(U)\frac{\partial}{\partial x}U = 0, \tag{7.39}$$

where U is a vector, and A is a matrix. We assume this is a hyperbolic equation. We study a small disturbance around a constant solution U_0:

$$U = U_0 + \epsilon U_1 + \cdots . \tag{7.40}$$

Accordingly, A is also expanded as

$$A(U)=A_0+\epsilon U_1\cdot\nabla_U A_0+\cdots. \tag{7.41}$$

To order ϵ

$$LU_1=0, \tag{7.42}$$

where

$$L\equiv\frac{\partial}{\partial t}+A_0\frac{\partial}{\partial x}, \tag{7.43}$$

and the next order is

$$LU_2=-(U_1\cdot\nabla_U A_0)\frac{\partial U_1}{\partial x}. \tag{7.44}$$

Equation (7.42) is essentially a wave equation. Let us assume that the eigenvalues of the matrix A_0 be real and distinct and we rewrite A_0 as $\Sigma_j v_j R_j L_j$ in terms of its left eigenvector L_j and right eigenvector R_j such that $L_j A_0=v_j L_j$, and $A_0 R_j=v_j R_j$, respectively. Then the general solution to Eq. (7.42) reads

$$U_1=\sum_j h_j(x-v_j t)R_j. \tag{7.45}$$

Here, h_j are determined by the initial condition, so h_j (initial wave form) is modifiable (i.e., we may renormalize it).

Since all the velocities of the simple waves (components) are distinct, far away from the source we may ignore the interferences among these waves. Therefore let us consider only h_j as a representative. U_2 contains a resonant term for each component. Hence Eq. (7.44) reads [step (1)]

$$\begin{aligned}LU_2&=-h_j(x-v_j t)h_j'(x-v_j t)(R_j\cdot\nabla_U)A_0R_j\\&=F(h(x-v_j t)).\end{aligned} \tag{7.46}$$

It is obvious that a secular contribution arises from $F(h(x-v_j t))$ as can be seen easily from the comoving coordinates. The usual subtraction and absorption into U_1 can be done for each component as [step (2)]

$$\begin{aligned}(U(t,x)-U_0)/\epsilon=h_{jR}(\tau,\xi)R_j+\epsilon(&U_2(t,x)\\&-\hat{U}_2(\tau,\xi,x-v_j t)).\end{aligned} \tag{7.47}$$

Projection of Eq. (7.46) onto the eigenvector R_j gives [step (3)]

$$\mathcal{S}\hat{U}_2(\tau,\xi,x-v_j t)\equiv[L_j\cdot(\partial_\tau+A_0\partial_\xi)\hat{U}_2]=[F]_\parallel, \tag{7.48}$$

with

$$[F]_\parallel=-h_j(x-v_j t)h_j'(x-v_j t)L_j(R_j\cdot\nabla_U)A_0R_j. \tag{7.49}$$

Hence the proto-RG equation is [step (4)]

$$\left(\frac{\partial}{\partial\tau}+v_j\frac{\partial}{\partial\xi}\right)h_{jR}(\tau,\xi)+\epsilon[L_j(R_j\nabla_U A_0)R_j]h_{jR}\frac{\partial}{\partial\xi}h_{jR}=0, \tag{7.50}$$

This is also the RG equation for this case.

The method to construct a proto-RG operator may be summarized as follows: Let $L(\partial_x)u=0$ be the unperturbed linear equation, and u_0 is its solution (of our interest). Here x collectively denotes all the independent variables. Then, split the differential operators as $\partial/\partial x\to\partial/\partial x+\partial/\partial\chi$, introduce the results into L and subtract the original operator from this to make $\mathcal{S}'=L(\partial_x+\partial_\chi)-L(\partial_x)$. In this procedure all the functions appearing in L must be moved beforehand to the left of differential operators, calculating needed derivatives [e.g., $(\partial/\partial t)f(t)\to f'(t)+f(t)\partial/\partial t$]. Then, apply $\mathcal{S}'$ to the perturbation result with secular terms, and project the result onto the null space of $L(\partial_x)$. Finally, identify χ and x in the result. The overall result defines the proto RG operator $\mathcal{S}$ [as exemplified in Eq. (3.45)].

VIII. NEWELL-WHITEHEAD EQUATION AND ITS GENERALIZATION

We consider here the two-dimensional Swift-Hohenberg equation widely used as a simple model of the Rayleigh-Bénard convection [26],

$$\frac{\partial u}{\partial t}=\epsilon u-u^3-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)^2 u, \tag{8.1}$$

where ϵ is a control parameter or a reduced Rayleigh number, a measure of the degree of convective instability of the quiescent state $u=0$. For small positive ϵ, the system exhibits a supercritical bifurcation. Since we wish to treat $\epsilon u-u^3$ as a perturbative term, to be consistent ϵu and u^3 must be of the same order. We scale u as $\sqrt{\epsilon}u$, and denote the new u with the same symbol. Then, the original equation reads

$$\frac{\partial u}{\partial t}=\epsilon(u-u^3)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)^2 u. \tag{8.2}$$

We consider this on the whole plane for all positive t. As a zeroth order solution, we choose the roll solution along the y axis: $Ae^{ikx}+$c.c., where A is a complex numerical constant. We expand u around this solution as

$$u=Ae^{ikx}+A^*e^{-ikx}+\epsilon u_1+\epsilon^2 u_2+\cdots+\text{c.c.} \tag{8.3}$$

The first order correction obeys [step (1)]

$$\frac{\partial u_1}{\partial t}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)^2 u_1=(1-3|A|^2)Ae^{ikx}-A^3e^{3ikx}. \tag{8.4}$$

This is a linear PDE, so we may write the solution in the following form [step (2)]:

$$u_1=P_1(t,\boldsymbol{r})e^{ikx}+Q_1(t,\boldsymbol{r})e^{3ikx}, \tag{8.5}$$

where $\boldsymbol{r}=(x,y)$. Since e^{ikx} is a null solution to the linear operator, P_1 must contain spatially secular (that is, unbounded or not integrable) terms.

Since

$$\left[\frac{\partial}{\partial t}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)^2\right]P_1 e^{ikx}=(1-3|A|)^2)Ae^{ikx}, \tag{8.6}$$

we have [step (3)]

$$\left[\frac{\partial}{\partial t}+\frac{\partial^4}{\partial x^4}+4ik\frac{\partial^3}{\partial x^3}+2\frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2}+\left(-4k^2\frac{\partial^2}{\partial x^2}+4ik\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial x}+\frac{\partial^4}{\partial y^4}\right)\right]P_1 \equiv LP_1=(1-3|A|)^2A. \tag{8.7}$$

Similarly, we can obtain

$$\left[\frac{\partial}{\partial t}+\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}+6ik\frac{\partial}{\partial x}-8k^2\right)^2\right]Q_1\equiv RQ_1=-A^3. \tag{8.8}$$

From this, we clearly see that P_1 can never be a constant, but Q_1 can.

The renormalized perturbation series reads

$$u=A_R(\tau,\boldsymbol{\rho})e^{ikx}+\epsilon[P_1(t,\boldsymbol{r})-P_1(\tau,\boldsymbol{\rho})]e^{ikx}+Q_1e^{3ikx}\cdots+\text{c.c.} \tag{8.9}$$

Hence the proto-RG equation reads [step (4)]

$$\left(\frac{\partial}{\partial\tau}+L_{\tau,\boldsymbol{\rho}}\right)A_R(\tau,\boldsymbol{\rho})=\epsilon(1-3|A_R|^2)A_R, \tag{8.10}$$

where $L_{\tau,\boldsymbol{\rho}}$ is L with the replacement $t\to\tau$, $\boldsymbol{r}\to\boldsymbol{\rho}$. It should be seen easily that to obtain this result, we may follow the more abstract procedure in the preceding subsections (or as is noted in Sec. III).

L contains much more terms than the standard result. To reduce the equation further, we must choose the way we observe the system. If we choose the order of the variables as $\partial/\partial t\sim\partial^2/\partial x^2\sim\partial^4/\partial y^4\sim\epsilon$ (i.e., $t\sim\epsilon^{-1}$, $x\sim\epsilon^{-1/2}$, and $y\sim\epsilon^{-1/4}$), then [step (5)]

$$\left(\frac{\partial}{\partial t}-4k^2\frac{\partial^2}{\partial x^2}+4ik\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial x}+\frac{\partial^4}{\partial y^4}\right)A_R(t,\boldsymbol{r})=\epsilon(1-3|A_R|^2)A_R, \tag{8.11}$$

the usual result. An important observation is that there is no other consistent choice if we wish to avoid a simple diffusion equation to this order. For example, if we wish to choose $\partial^4/\partial y^4\sim\partial^4/\partial y^2\partial x^2\sim(\partial/\partial t)\sim\epsilon$, then $\partial^2/\partial x^2$ and $\partial^3/\partial x\partial y^2$ dominate the lhs and cannot balance the order ϵ term on the rhs. Therefore Eq. (8.11) is the only consistent nontrivial result to order ϵ.

To obtain the next order, we need the equation for the second order term:

$$\left[\frac{\partial u_2}{\partial t}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)^2\right]u_2=\{(1-6|A|^2)P_1-3A^2P_1^*-3A^{*2}Q_1\}e^{ikx}+\{(1-6|A|^2)Q_1-3A^2P_1\}e^{3ikx}-3A^2Q_1e^{5ikx}. \tag{8.12}$$

The general form of the solution (the special solution) is given by

$$u_2=P_2(t,\boldsymbol{r})e^{ikx}+Q_2(t,\boldsymbol{r})e^{3ikx}+R_2e^{5ikx}. \tag{8.13}$$

Here, P_2 is obviously secular. Q_2 is also secular because the inhomogeneous term with e^{3ikx} is already secular in Eq. (8.12).

$$LP_2=(1-6|A|^2)P_1-3A^2P_1^*-3A^{*2}Q_1, \tag{8.14}$$

but note that we need only the last term, because P_1 is not constant. From Eq. (8.8) we obtain

$$Q_1=-\frac{A^3}{64k^4}. \tag{8.15}$$

Hence the proto-RG equation reads

$$LA_R=\epsilon(1-3|A_R|^2)A_R+\epsilon^2\frac{3}{64k^4}|A_R|^4A_R. \tag{8.16}$$

Here, we have used the fact that the terms containing e^{3ikx}, e^{5ikx}, etc., may be ignored, and that we may ignore all the terms explicitly dependent on space time due to the space-time translational symmetry of the system [27]. This is actually the RG equation to order ϵ^2.

Notice that Eq. (8.10) is the equation derived by Graham [9]. It is now clear that it is not a consistent equation to order ϵ. Up to order $\epsilon^{3/2}$, we can make Eq. (8.10) consistent by dropping $\partial^4/\partial x^4$ in L [28]. If we wish to retain all the differential operators in Eq. (8.10), we need an additional inhomogeneous term as in Eq. (8.16).

Let us apply the proto-RG operator scheme to the second order result. Its merit is that we need not assume the form such as Eqs. (8.5) or (8.13). From Eq. (8.12) the operator reads

$$\mathcal{S}\equiv P\left\{\frac{\partial}{\partial\tau}+\left[\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial x}\right)^2+\left(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial y}\right)^2+k^2\right]^2-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)^2\right\}, \tag{8.17}$$

where P is the projection operator extracting the coefficient of e^{ikx} (subsequent replacement of the variables such as $x\to\xi$ is implied).

$$\hat{\mathcal{S}u}_2=[(1-6|A|^2)u_1-(3A^2e^{2ikx}+A^{*2}e^{-2ikx})u_1]_{\|}, \quad (8.18)$$

where $[\ ]_{\|}$ means the coefficient of e^{ikx} (and then the independent variables are all replaced accordingly). Applying the proto-RG operator to the renormalized perturbation, we have

$$\left\{\frac{\partial}{\partial\tau}+\left[\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial x}\right)^2+\left(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial y}\right)^2+k^2\right]^2 -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)^2\right\}A_R(\tau,\boldsymbol{\rho})e^{ikx}=\cdots. \quad (8.19)$$

We immediately see this to be equal to $e^{ikx}(\partial/\partial\tau+L_{\tau,\boldsymbol{\rho}})A_R=\cdots$. We need the nonsecular terms in u_1 to write the right hand side of Eq. (8.18) explicitly. We must look into the detail somewhat. Equation (8.4) tells us that (i) the term proportional to $e^{\pm ikx}$ is secular, and (ii) the term proportional to $e^{\pm 3ikx}$ is nonsecular, so we look for the term of the form $Q_1e^{\pm 3ikx}$. This gives $64k^4QA_1=-A^3$ (or $-A^{*3}$). Hence the term proportional to e^{ikx} in Eq. (8.18) reads $+3k^4A|A|^4/64$. Therefore the result agrees with the one already obtained.

IX. CONCLUDING REMARKS

The key element of singular perturbation is to separate out global systematic effects of perturbation. If we can derive equations governing these effects, then the most important results of singular perturbation can be obtained. These equations are reduced equations such as slow-motion equations derived traditionally by reductive perturbation methods. Renormalization-group equations in our approach are the equations governing the slow changes of the parameters that are integration constants of unperturbed problems. This is why the RG approach naturally gives the slow motion equation and can reproduce results of the so-called reductive perturbations.

If the equation of motion is known of the parameters that are slowly modified by the secular effects of perturbations, the global (e.g., long term) behavior of the perturbed system is completely known. Therefore reductive perturbation is conceptually the key element of singular perturbation. In this paper we have shown how to directly obtain RG equations (i.e., the results of reductive perturbations) without knowing explicit singular perturbation results. Consequently, reductive perturbation (realized by RG methods) has become the key, not only in principle but in practice, to singular perturbation. Besides we wish to point out that the RG equation facilitates rigorous error bounds [6].

The reader might ask how general the reductive RG is. For example, in the boundary layer type problems reductive perturbation is seldom mentioned. In our RG approach, boundary layer type problems are solved without matching from the inner expansion [8]. Therefore these problems are equivalent to problems of long term asymptotics. Thus reductive perturbation becomes meaningful even for boundary layer problems in our approach. Therefore if we use the traditional terminologies, we conclude that reductive perturbation is the key to singular perturbation.

In this paper, we have largely freed the RG theoretical reduction from explicit perturbation results with the aid of the proto-RG equation. For example, to the lowest nontrivial order, that is, to the order many famous phenomenological equations are obtained, we do not need any explicit result. The traditional reductive perturbation already has this feature, so this may not be surprising. Now we can combine this advantage and the advantages of our RG procedures illustrated in Ref. [8]. As is explained in Ref. [6], reductive perturbation sets up a function space (usually an L_2 space) to solve a given problem and the condition to force the solution into this space (the solvability condition) gives the reduced equation. In contrast, the reductive RG constructs a solution in a much wider function space, and then later trims the solution by renormalization to fit in a certain conventional function space. Therefore reductive RG is conceptually more natural (less constrained), and is expected to be more versatile. However, we have not been able to make any relevant mathematical statement, because the general idea of doing functional analysis without setting up function spaces is remote from the current functional analytic practice.

RG, especially the field theoretical RG, is a method to extract structurally stable results against alteration of microscopic details. As has been clearly demonstrated in Ref. [29] or summarized in Ref. [6], the mathematical structure of the perturbative RG applied to differential equations and that applied to field theory are identical. Since asymptotic analysis is a method to discard ''nonasymptotic details,'' asymptotic analysis is almost tautologically a pursuit of structural stability in a certain sense. What sort of structural stability we should pursue depends on the problem. The analogy between the field theoretical RG and study of long time behavior has told us that to extract stable features against perturbing initial conditions (or short time behaviors) is the asymptotic analysis of long time behavior. Our statistical mechanical RG experience strongly suggests that all the asymptotic analyses can be cast into the RG theoretical form. This is the reason that we feel the RG approach is much more general and more powerful than we have already experienced. It is desirable to demonstrate its generality theoretically, not with various examples.

ACKNOWLEDGMENTS

Y.O. is grateful to Fred Furtado, Yasu Shiwa, and Sin-ichi Sasa for useful discussions, corrections, and correspondences, and to Yoichiro Takahashi for his hospitality at the Research Institute for Mathematical Sciences in Kyoto and for a partial financial aid. The present work is, in part, supported by the National Science Foundation, Grant No. NSF-DMR 99-70690, and by the Japan Society for Promotion of Science, Grant-in-Aid for Scientific Research (A) 09304022 (representative: Y. Takahashi).

[1] L.-Y. Chen, N. Goldenfeld, Y. Oono, and G. Paquette, Physica A **204**, 111-133 (1994).
[2] T. Taniuti, Prog. Theor. Phys. Suppl. **55**, 1 (1974).
[3] H. Mori and Y. Kuramoto, *Dissipative Structure and Chaos*, translated by G. C. Paquette (Springer, New York, 1998).
[4] O. Pashko and Y. Oono, Int. J. Mod. Phys. B **41**, 3290 (2000).
[5] For weakly nonlinear oscillators and for the cases of amplitude equations rigorous justification of the RG procedure is available. See Refs. [6] and [7].
[6] An introductory review of RG applied to differential equations from this point of view will appear in the Gakushuin International Summer School of Mathematical Physics special issue of Int. J. Mod. Phys. B **14**, 1327 (2000).
[7] M. Ziane, J. Math. Phys. **41**, 3290 (2000).
[8] L.-Y. Chen, N. Goldenfeld, and Y. Oono, Phys. Rev. E **54**, 376 (1996).
[9] R. Graham, Phys. Rev. Lett. **76**, 2185 (1996); **80**, 3888(E) (1998).
[10] A. C. Newell and J. A. Whitehead, J. Fluid Mech. **38**, 279 (1969).
[11] G. H. Gunaratne, Qi Ouyang, and H. L. Swinney, Phys. Rev. E **50**, 2802 (1994).
[12] K. Matsuba and K. Nozaki, Phys. Rev. Lett. **80**, 3886 (1998).
[13] S. Sasa, Physica D **108**, 45 (1997).
[14] In this calculation the difference between t and τ has been assumed to be a higher order infinitesimal than the powers of ϵ. This is a simplified method; we can also assume that t and τ are close but not infinitesimally close as in Ref. [8]. Under this condition, Z_2 depends not only on τ but also on t.
[15] We can characterize resonant and nonresonant singular terms more generally as follows. The resonant singular term comes from the solution to $Lu=f$, where L is the linear operator in the perturbation equation and f satisfies $Lf=0$. Nonresonant singular terms arise in the solution with f being singular (e.g., not L_2) but not $Lf=0$.
[16] S. Goto, Y. Masutomi, and K. Nozaki, Prog. Theor. Phys. **102**, 471 (1999).
[17] M. Nishiura, *Nonlinear Problems I* (Iwanami, Tokyo, 1999) (in Japanese).
[18] We do not explicitly write nonresonant secular terms, exploiting the observation (1) in Sec. II.
[19] We have recently realized that Estrada has been advocating the expansion in terms of δ functions and its derivatives. See L. G. Hernández-Urená and R. Estrada, J. Math. Anal. Appl. **191**, 40 (1995).
[20] A. Voros, Ann. Inst. Henri Poincaré Phys. Theor. **39**, 211 (1983); T. Kawai and Y. Takei, *Algebraic Analysis of Singular Perturbation* (Iwanami, Tokyo, 1998); see also J. G. Byatt-Smith, Stud. Appl. Math. **103**, 339 (1999).
[21] Here, by ''general solution'' we mean any solution in the set of all appropriately differentiable functions satisfying Eq. (6.1). This set contains many special cases, but we wish to discuss the properties that hold for any element in this set.
[22] The use of the relation in the orthogonal complement is unusual, but it is understood as a shortcut. Because the original equality and the equality in the null space automatically implies the equality in the orthogonal complement, the relation we have used should not impose any extra condition. However, the same condition may be more easily seen from one subspace than the other. The following more honest treatment makes the point clearer.
[23] S. M. Allen and J. W. Cahn, Acta Metall. **27**, 1085 (1979).
[24] K. Kawasaki and T. Ohta, Prog. Theor. Phys. **67**, 147 (1982).
[25] Y. Oono, in *Mathematics of Microstructure Evolution*, edited by L-Q. Chen, B. Fultz, J. W. Cahn, J. R. Manning, J. E. Morral, and J. Simmons (TMS SIAM, Warrendale, PA, 1996), p. 355.
[26] J. Swift and P. C. Hohenberg, Phys. Rev. A **15**, 319 (1977).
[27] The logic here is quite parallel to the case of weak nonlinear oscillators discussed in Sec. III.
[28] K. Matsuba and K. Nozaki, Phys. Rev. E **56**, R4926 (1997).
[29] N. D. Goldenfeld, O. Martin, and Y. Oono, J. Sci. Comput. **4**, 355 (1989).